Free vibration analysis of a rotating Timoshenko beam by differential transform method

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Abstract
Purpose – To perform the flapwise bending vibration analysis of a rotating cantilever Timoshenko beam.
Design/methodology/approach – Kinetic and potential energy expressions are derived step by step. Hamiltonian approach is used to obtain the governing equations of motion. Differential transform method (DTM) is applied to solve these equations.
Findings – It is observed that the $\rho I \Omega^2 \theta$ term which is ignored by many researchers and which becomes more important as the rotational speed parameter increases must be included in the formulation.
Originality/value – Kinetic and potential energy expressions for rotating Timoshenko beams are derived clearly step by step. It is the first time, for the best of author's knowledge, that DTM has been applied to the blade type rotating Timoshenko beams.

Keywords Rotation measurement, Transforms, Cantilevers

Paper type Research paper

1. Introduction

There has been a growing interest in the analysis of the free vibration characteristics of elastic structures that rotate with a constant angular velocity. Numerous structural configurations such as turbine, compressor and helicopter blades, spinning spacecraft and satellite booms fall into this category.

Investigation of the natural frequency variation of rotating beams originated from the work of Southwell and Gough (1921). They suggested a simple equation (known as the Southwell equation), which is based on the Rayleigh energy theorem to estimate the natural frequencies of rotating cantilever beams. Liebers (1930) and Theodorsen (1950) extended Southwell's work. Earlier studies mainly focused on Euler Bernoulli beams. However, due to the inclusion of shear deformation and rotary inertia effects, Timoshenko beam theory is more accurate than Euler Bernoulli beam theory. Therefore, considerable research has been carried out on the free vibrations of rotating Timoshenko beams, recently (Stafford and Giurgiutiu, 1975; Yokoyama, 1988; Lee and Kuo, 1993; Du *et al.*, 1994; Nagaraj, 1996; Bazoune *et al.*, 1999; Banerjee, 2001).

Different types of solution procedures, i.e. the finite element method, the Frobenius method of series, the Galerkin method, the Myklestad procedure, the finite differences approach, the perturbation technique, Bessel functions, etc. may be found in the literature. In this study, the differential transform method (DTM), which is a semi analytical-numerical technique that depends on Taylor series expansion and which was introduced by Zhou (1986) in his study about electrical circuits, is used. Recently, Banerjee (2001) has developed the Dynamic Stiffness Method for a rotating cantilever Timoshenko beam that is based on Frobenius series expansion and claims its superiority of finding more correct results. However, application of this method, as he pointed out, is not so easy. On the other hand, the advantage of the DTM is its simplicity and high accuracy.

2. Description of the problem

In Figure 1, a cantilever beam of length L, which is rigidly mounted on the periphery of a rigid hub of radius R, is shown. The hub rotates about its axis at a constant angular speed Ω. The origin is taken to be at the left-hand end of the beam. The x-axis coincides with the neutral axis of the beam in the undeflected position, the z-axis is parallel to the axis of rotation (but not coincidental) and the y-axis lies in the plane of rotation. The beam considered here is doubly symmetric such that the mass axis, the centroidal axis and the elastic axis are coincident.

The following assumptions are made in this study:
- The out-of-plane displacement of the beam is small.
- The cross sections that are initially perpendicular to the neutral axis of the beam remain plane, but no longer perpendicular to the neutral axis during bending.
- The beam material is homogeneous and isotropic.
- Coriolis effects are not included.

3. Formulation

Most of the investigators begin their studies by introducing the equations of motion directly. However, in this paper the potential and the kinetic energy expressions are derived step by step and then, equations of motion are obtained using the Hamiltonian approach.

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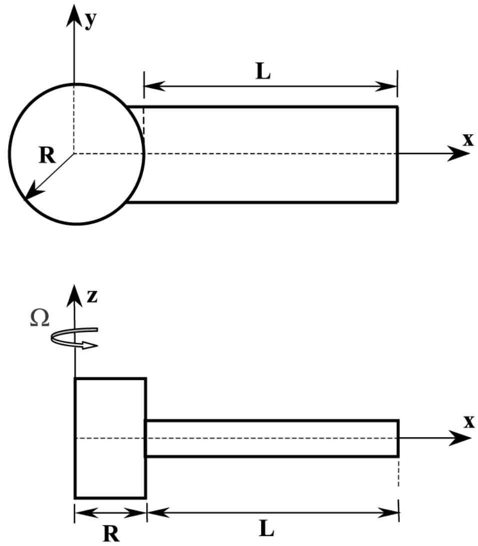
Figure 1 Configuration of a rotating cantilever Timoshenko beam



3.1 Strain displacement relations

Several different classical definitions of strain may be found in literature, depending on the mathematical formulation, reference states (based on deformed or undeformed positions), and coordinate systems used.

The flapwise motion of a Timoshenko beam is given in Figure 2(a) and (b). In Figure 2(a), longitudinal view of displacement of a point is introduced and in Figure 2(b), cross-sectional view of displacement of the same point is introduced. In these graphics, the point is represented by P_0 before deformation and by P after deformation.

The vector position of the point is defined as:

$$\vec{r} = x\vec{\mathrm{i}} + y\vec{\mathrm{j}} + z\vec{\mathrm{k}} \tag{1}$$

Keeping in mind that the subscripts $(\,)_0$ and $(\,)_1$ represent the positions of the point before and after deformation, respectively, these positions can be given as follows.

- Before deformation:

$$x_0 = R + x \tag{2a}$$

$$y_0 = \eta \tag{2b}$$

$$z_0 = \xi \tag{2c}$$

- After deformation:

$$x_1 = R + x + u_0 - \xi\theta \tag{3a}$$

$$y_1 = \eta \tag{3b}$$

$$z_1 = w + \xi \tag{3c}$$

Here, x is the spanwise distance of the point from the hub edge, u_o is the axial displacement due to the centrifugal force, η is the transverse distance of the point from the axis of rotation, ξ is the vertical distance of the point from the middle plane, w is the bending displacement and θ is the rotation due to bending.

Knowing that $\vec{r}_0$ and $\vec{r}_1$ are the vector positions of the point on the undeformed and deformed blade, respectively, the position vector differentials can be given by:

$$\mathrm{d}\vec{r}_0 = \mathrm{d}x\vec{i} + \mathrm{d}\eta\vec{j} + \mathrm{d}\xi\vec{k} \tag{4}$$

$$\mathrm{d}\vec{r}_1 = [(1 + u_0' - \xi\theta')\mathrm{d}x - \theta\,\mathrm{d}\xi]\vec{i} + \mathrm{d}\eta\vec{j} + (w'\,\mathrm{d}x + \mathrm{d}\xi)\vec{\mathrm{k}} \tag{5}$$

where $\mathrm{d}x$, $\mathrm{d}\eta$ and $\mathrm{d}\xi$ are the increments along the deformed elastic axis and two cross-sectional axes, respectively.

The classical strain tensor ε_{ij} in terms of $\vec{r}_1$ and $\vec{r}_0$ may be expressed as follows:

$$\mathrm{d}\vec{r}_1 \cdot \mathrm{d}\vec{r}_1 - \mathrm{d}\vec{r}_0 \cdot \mathrm{d}\vec{r}_0 = 2\lfloor\, \mathrm{d}x \;\; \mathrm{d}\eta \;\; \mathrm{d}\xi \rfloor [\varepsilon_{ij}] \begin{Bmatrix} \mathrm{d}x \\ \mathrm{d}\eta \\ \mathrm{d}\xi \end{Bmatrix} \tag{6}$$

Substituting equations (4) and (5) into equation (6), components of the strain tensor ε_{ij} are obtained as:

$$2\varepsilon_{xx} = (1 + u_0' - \xi\theta')^2 + (w')^2 - 1 \tag{7a}$$

$$\gamma_{x\eta} = 0 \tag{7b}$$

$$\gamma_{x\xi} = w' - (1 + u_0' - \xi\theta')\theta \tag{7c}$$

In order to obtain simple expressions for the strain components, higher order terms should be neglected, so an order of magnitude analysis is necessary. The ordering scheme is taken from Hodges and Dowell (1974) and introduced in Table I. Simplified strain components are obtained as follows by ignoring the terms which are higher than ε^2.

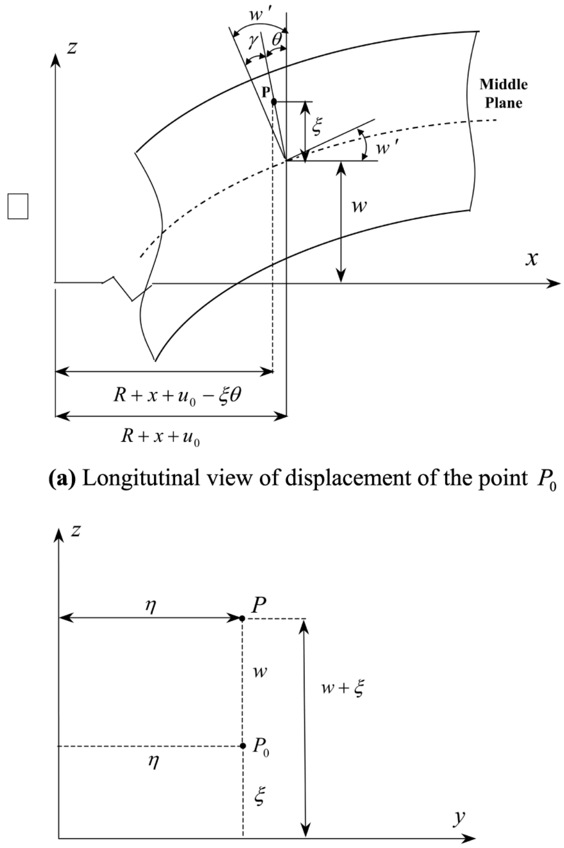
$$\varepsilon_{xx} = u_0' - \xi\theta' + \frac{(w')^2}{2} \tag{8a}$$

$$\gamma_{x\eta} = 0 \tag{8b}$$

$$\gamma_{x\xi} = w' - \theta \tag{8c}$$

where $\gamma_{x\xi}$ is the loss of slope that is equal to the shear strain.

Figure 2



(a) Longitutinal view of displacement of the point P_0

(b) Cross sectional view of displacement of the point P_0

Table I Ordering scheme

$\frac{u}{R} = O(\varepsilon^2)$	$\frac{x}{R} = O(1)$
$\frac{v}{R} = O(\varepsilon)$	$\frac{\eta}{R} = O(\varepsilon)$
$\frac{w}{R} = O(\varepsilon)$	$\frac{\xi}{R} = O(\varepsilon)$
$\Psi = O(\varepsilon)$	$\gamma = w' - \varphi = O(\varepsilon^2)$
$\varphi = O(\varepsilon)$	

3.2 Expression for the potential energy

The strain energy due to bending, U_b, is given by:

$$U_b = \iiint_V \frac{E\varepsilon_{xx}^2}{2} \mathrm{d}V \tag{9}$$

Substituting equation (8a) into equation (9) leads to:

$$U_b = \frac{E}{2} \int_A \int_0^L \left\{ u_0' - \xi\theta' + \frac{1}{2}(w')^2 \right\}^2 \mathrm{d}x\, \mathrm{d}A \tag{10}$$

where A is the cross-sectional area and E is the Young's modulus. Taking integration over the blade cross section gives:

$$U_b = \frac{1}{2}\int_0^L EA(u_0')^2 \mathrm{d}x + \frac{1}{2}\int_0^L EI(\theta')^2 \mathrm{d}x + \frac{1}{2}\int_0^L EAu_0'(w')^2 \mathrm{d}x \tag{11}$$

where:

$$I = \int_A \xi^2 \mathrm{d}A$$

is the second moment of area of the beam cross section about the y-axis, EI is the bending rigidity and EA is the axial rigidity of the beam cross section.

The uniform strain ε_o and the associated axial displacement u_o due to the centrifugal force, $T(x)$ is given by:

$$u_0'(x) = \varepsilon_0(x) = \frac{T(x)}{EA} \tag{12}$$

where the centrifugal force that varies along the spanwise direction of the beam is:

$$T(x) = \int_x^L \rho A \Omega^2 (R + x) \mathrm{d}x \tag{13}$$

Substituting equation (13) into equation (12) and noting that the

$$\frac{1}{2}\int_0^L \frac{T^2(x)}{EA}\mathrm{d}x$$

term is constant and will be denoted as C_1, we get the final form of the bending strain energy as follows:

$$U_b = \frac{1}{2}\int_0^L EI(\theta')^2\mathrm{d}x + \frac{1}{2}\int_0^L T(w')^2\mathrm{d}x + C_1 \tag{14}$$

The strain energy due to shear, U_s, is given by:

$$U_s = \iiint_V \frac{G\left(\gamma_{x\eta}^2+\gamma_{x\xi}^2\right)}{2}\,\mathrm{d}V = \frac{1}{2}\int_A\int_0^L G\left(\gamma_{x\eta}^2+\gamma_{x\xi}^2\right)\mathrm{d}A\,\mathrm{d}x \tag{15}$$

where G is the shear modulus.

Substituting equations (8b) and (8c) into equation (15), we get the final form of the shear strain energy as follows:

$$U_s = \frac{1}{2}\int_A\int_0^L G(w'-\theta)^2\mathrm{d}A\,\mathrm{d}x = \frac{1}{2}\int_0^L kAG(w'-\theta)^2\mathrm{d}x \tag{16}$$

where k is the shear correction factor that depends on the shape of the cross section (for circular and rectangular cross sections, the values of k are 2/3 and 3/4, respectively) and that is used to take into account the variation of shear stress across the thickness and kAG is the shear rigidity.

Combining all the strain energy components, the total strain energy U of the beam is found to be:

$$U = U_b + U_s = \frac{1}{2}\left[\int_0^L \{EI(\theta')^2 + kAG(w'-\theta)^2 + T(w')^2\}\mathrm{d}x\right] \tag{17}$$

The first term in equation (17) represents the flexural strain energy, the second term the shear strain energy and the last term the strain energy due to the centrifugal force.

3.3 Expression for the kinetic energy

The velocity of the representative point P due to rotation of the beam is given by:

$$\vec{V} = \frac{\delta\vec{r}}{\delta t} + \Omega\vec{\mathrm{k}}\times\vec{r}_1 \tag{18}$$

The total velocity $\vec{V}$ can be expressed in terms of deformed positions as follows:

$$\vec{V} = (\dot{x}_1 - \Omega y_1)\vec{i} + (\dot{y}_1 + \Omega x_1)\vec{j} + \dot{z}_1\vec{k} \tag{19}$$

Substituting the derivatives of equations (3a)-(3c) with respect to time, t, into equation (19), the velocity components are obtained as follows:

$$V_x = -\xi\dot{\theta} - \eta\Omega \tag{20a}$$

$$V_y = (R + x + u_0 - \xi\theta)\Omega \tag{20b}$$

$$V_z = \dot{w} \tag{20c}$$

The kinetic energy is given by:

$$\Im = \frac{1}{2}\int_A\int_0^L (V_x^2 + V_y^2 + V_z^2)\rho\,\mathrm{d}A\,\mathrm{d}x \tag{21}$$

Substituting the velocity components introduced in equations (20a)-(20c) into equation (21), the final form of the kinetic energy expression is obtained.

$$\Im = \frac{1}{2}\int_0^L [\rho A\dot{w}^2 + \rho I\dot{\theta}^2 + \rho I\Omega^2\theta^2]\mathrm{d}x \tag{22}$$

3.4 Derivation of the governing differential equations of motion

The governing differential equations of motion and the boundary conditions can be derived by means of the Hamiltonian approach, which can be stated in the following form:

$$\int_{t_1}^{t_2} (\delta\Im - \delta U)\mathrm{d}t = 0 \tag{23}$$

where $\delta w = \delta\theta = 0$ at t_1 and t_2.

After integration, the equations of motion are obtained as follows:

$$-\rho A\ddot{w} + (Tw')' + [kAG(w'-\theta)]' = 0 \tag{24}$$

$$-\rho I\ddot{\theta} + \rho I\Omega^2\theta + (EI\theta')' + kAG(w'-\theta) = 0 \tag{25}$$

Equations (24) and (25) define completely the free vibration of a uniform rotating Timoshenko beam. Here w is the out-of-plane displacement and θ is the rotation due to bending.

It must be noted that although the term $\rho I\Omega^2\theta$ can be important when the constant rotational speed, Ω is high, it is not taken into account by some authors. The physical description of this term is that as a result of the bending deformation, the radii of the elements that are symmetrically placed with respect to the mid-plane of the beam cross section are different so these elements have different centrifugal forces although the net centrifugal force is independent of the section rotation. Thus, a moment that has the value of $\rho I\Omega^2\theta$ appears.

Primes in equations (24) and (25) mean differentiation with respect to the spanwise position, x and dots mean differentiation with respect to time, t; ρ is the material density and ρA is the mass per unit length. Here, ρA and ρI are the inertia terms.

The boundary conditions at $x = 0$ and $x = L$ for equations (24) and (25) are given by:

$$\theta'\delta\theta|_0^L = 0 \tag{26}$$

$$[Tw' + kAG(w'-\theta)]\delta w|_0^L = 0 \tag{27}$$

3.5 Free vibration analysis

A sinusoidal variation of $\nu(x,t)$ and $\theta(x,t)$ with circular frequency ω can be given by:

$$w(x,t) = \overline{W}(x)\mathrm{e}^{\mathrm{i}\omega t} \tag{28}$$

$$\theta(x,t) = \bar{\theta}(x)\mathrm{e}^{\mathrm{i}\omega t} \tag{29}$$

Substituting equations (28) and (29) into equations (24) and (25), equations of motion are expressed as follows:

$$\rho I\omega^2\bar{\theta}+\rho I\Omega^2\bar{\theta}+EI\bar{\theta}''+kAG(\overline{W}'-\bar{\theta})=0 \tag{30}$$

$$\mu\omega^2\overline{W}+(Tw')'+kAG(\overline{W}''-\bar{\theta}')=0 \tag{31}$$

3.6 Nondimensionalizing of the problem

The dimensionless parameters that are used to simplify the equations and to make comparisons with the studies in literature can be given as follows:

$$\xi=\frac{x}{L},\quad \delta=\frac{R}{L},\quad W(\xi)=\frac{\overline{W}}{L},\quad r^2=\frac{I}{AL^2},\quad s^2=\frac{EI}{kAGL^2},\quad \eta^2=\frac{\rho AL^4\Omega^2}{EI},\quad \mu^2=\frac{\rho AL^4\omega^2}{EI} \tag{32}$$

Here δ is the hub radius parameter, r is the rotary inertia parameter, s is the shear deformation parameter, η is the rotational speed parameter and μ is the frequency parameter.

Using the first two dimensionless parameters, dimensionless expression for the centrifugal force can be written as follows:

$$T(\xi)=\rho A\Omega^2L^2\left[\delta(1-\xi)+\frac{(1-\xi^2)}{2}\right] \tag{33}$$

Substituting equations (32) and (33) into equations (30) and (31), the dimensionless form of the equations of motion are obtained

$$\theta''+\eta^2r^2\left(1+\frac{\omega^2}{\Omega^2}\right)\theta+\frac{1}{s^2}(w'-\theta)=0 \tag{34}$$

$$\left\{\left[\delta(1-\xi)+\frac{(1-\xi^2)}{2}\right]w'\right\}'+\left(\frac{\omega^2}{\Omega^2}\right)w+\frac{1}{s^2\eta^2}(w''-\theta')=0 \tag{35}$$

4. Differential transform method

The DTM is one of the useful techniques to solve the ordinary differential equations with fast convergence rate and small calculation error. One advantage of this method over the integral transformation methods is its ability to handle nonlinear differential equations, either initial value problems or boundary value problems. It was introduced by Zhou (1986) in his study about electrical circuit. Recently, it has gained much attention by researchers (Arikoglu and Ozkol, 2004, 2005; Bert and Zeng, 2004; Chen and Ju, 2004; Ozdemir and Kaya, 2005).

Let $f(x)$ be analytic in a domain D and let $x=x_0$ represent any point in D. The function $f(x)$ is then represented by one power series whose center is located at x_0. The differential transform of the function $f(x)$ is defined as follows:

$$F[k]=\frac{1}{k!}\left(\frac{\mathrm{d}^kf(x)}{\mathrm{d}x^k}\right)_{x=x_0} \tag{36}$$

where $f(x)$ is the original function and $F[k]$ is the transformed function.

The inverse transformation is defined as:

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^kF[k] \tag{37}$$

Combining equations (36) and (37), one obtains:

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{\mathrm{d}^kf(x)}{\mathrm{d}x^k}\right)_{x=x_0} \tag{38}$$

Considering equation (38), it is noticed that the concept of differential transform is derived from Taylor series expansion. However, the method does not evaluate the derivatives symbolically.

In actual applications, the function $f(x)$ is expressed by a finite series and equation (38) can be written as follows:

$$f(x)=\sum_{k=0}^{n}\frac{(x-x_0)^k}{k!}\left(\frac{\mathrm{d}^kf(x)}{\mathrm{d}x^k}\right)_{x=x_0} \tag{39}$$

which implies that:

$$f(x)=\sum_{k=n+1}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{\mathrm{d}^kf(x)}{\mathrm{d}x^k}\right)_{x=x_0}$$

is negligibly small. In this study, the convergence of the natural frequencies determines the value of n.

Theorems that are frequently used in the transformation of the equations of motion and the boundary conditions are introduced in Tables II and III, respectively.

5. Solution with DTM

In the solution stage, the DTM is applied to the equations (34) and (35) by using the rules given in Table II and the following expressions are obtained.

$$\left(\delta+\frac{1}{2}+\frac{1}{s^2\eta^2}\right)(k+2)(k+1)W(k+2)-\delta(k+1)^2W(k+1)+\left[\frac{\omega^2}{\Omega^2}-\frac{k(k+1)}{2}\right]W(k)-(k+1)\theta(k+1)=0 \tag{40}$$

$$(k+2)(k+1)\theta(k+2)+\left[\eta^2r^2\left(1+\frac{\omega^2}{\Omega^2}\right)-\frac{1}{s^2}\right]\theta(k)+\frac{1}{s^2}(k+1)W(k+1)=0 \tag{41}$$

Applying the DTM to equations (26) and (27), the boundary conditions are given by:

$$\text{at }\ \xi=0\Rightarrow\theta(0)=W(0)=0 \tag{42}$$

$$\text{at }\ \xi=1\Rightarrow\sum_{k=0}^{\infty}k\theta(k)=0 \tag{43}$$

$$\sum_{k=0}^{\infty}[kW(k)-\theta(k)]=0 \tag{44}$$

Table II Basic theorems of DTM for equations of motion

Original function	Transformed functions
$f(x)=g(x)\pm h(x)$	$F[k]=G[k]\pm H[k]$
$f(x)=\lambda g(x)$	$F[k]=\lambda G[k]$
$f(x)=g(x)h(x)$	$F[k]=\sum_{l=0}^{k}G[k-l]H[l]$
$f(x)=\frac{\mathrm{d}^ng(x)}{\mathrm{d}x^n}$	$F[k]=\frac{(k+n)!}{k!}G[k+n]$
$f(x)=x^n$	$F[k]=\delta(k-n)=\begin{cases}0 & \text{if } k\neq n\\ 1 & \text{if } k=n\end{cases}$

Table III DTM theorems for boundary conditions

$x=0$		$x=1$	
Original BC	**Transformed BC**	**Original BC**	**Transformed BC**
$f(0)=0$	$F[0]=0$	$f(1)=0$	$\sum_{k=0}^{\infty} F[k]=0$
$\frac{df}{dx}(0)=0$	$F[1]=0$	$\frac{df}{dx}(1)=0$	$\sum_{k=0}^{\infty} kF[k]=0$
$\frac{d^2f}{dx^2}(0)=0$	$F[2]=0$	$\frac{d^2f}{dx^2}(1)=0$	$\sum_{k=0}^{\infty} k(k-1)F[k]=0$
$\frac{d^3f}{dx^3}(0)=0$	$F[3]=0$	$\frac{d^3f}{dx^3}(1)=0$	$\sum_{k=0}^{\infty} k(k-1)(k-2)F[k]=0$

In equations (40)-(44), $W(k)$ and $\theta(k)$ are the differential transforms of $w(\xi)$ and $\theta(\xi)$, respectively. Using equations (40) and (41), $W(k)$ and $\theta(k)$ values for $k=2,3,4\ldots$ can now be evaluated in terms of ω^2, c_1 and c_2. These values were achieved by using the Mathematica computer package. The results are introduced below for the values $\delta=0,\ r=0.04,\ s=2r$ and $\eta=8$.

$$\theta(2)=-78.125c_2$$

$$\theta(3)=-c_1(-4.40967+0.000267\omega^2)$$

$$\theta(4)=-c_2(174.47-0.045\omega^2)$$

$$W(2)=0.415007c_1$$

$$W(3)=-c_2(21.558+0.000885\omega^2)$$

$$W(4)=c_1(0.95-0.00024\omega^2)$$

where c_1 and c_2 are constants. Substituting all $W(i)$ and $\theta(i)$ terms into boundary condition expressions, i.e. equations (42) and (43), the following equation is obtained.

$$A_{j1}^{(n)}(\omega)c_1+A_{j2}^{(n)}(\omega)c_2=0,\ \ j=1,2,3,\ldots n \qquad (45)$$

where $A_{j1}^{(n)}(\omega), A_{j2}^{(n)}(\omega)$ are polynomials of ω corresponding to nth term.

When equation (45) is written in matrix form, we get:

$$\begin{bmatrix} A_{11}^n(\omega) & A_{12}^n(\omega) \\ A_{21}^n(\omega) & A_{22}^n(\omega) \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (46)$$

The eigenvalue equation is obtained from equation (46) as follows:

$$\begin{vmatrix} A_{11}^n(\omega) & A_{12}^n(\omega) \\ A_{21}^n(\omega) & A_{22}^n(\omega) \end{vmatrix}=0 \qquad (47)$$

Solving equation (47), we get $\omega=\omega_j^{(n)}$ where $j=1,2,3,\ldots n$. Here, $\omega_j^{(n)}$ is the jth estimated eigenvalue corresponding to n. The value of n is obtained by the following equation:

$$\left|\omega_j^{(n)}-\omega_j^{(n-1)}\right|\leq\varepsilon \qquad (48)$$

where ε is the tolerance parameter.

If equation (48) is satisfied, then we have jth eigenvalue $\omega_j^{(n)}$. In general, $\omega_j^{(n)}$ are conjugated complex values, and can be written as $\omega_j^{(n)}=a_j+ib_j$. Neglecting the small imaginary part bj, we have the jth natural frequency a_j.

6. Results and discussion

The computer package Mathematica is used to code the expressions obtained by using DTM, to calculate the natural frequencies and to plot the related graphics. In order to validate the computed results, an illustrative example taken from Banerjee (2001) is solved and the results are compared with the ones in the same reference paper.

The results of Table IV illustrate the effect of the rotational speed parameter, η, on the fundamental natural frequency of the Timoshenko beam. Present study and Banerjee (2001) show good agreement up to the fourth digit. However, the results of Lee and Kuo (1993) differ from the results of this study and the difference increases with the increasing rotational speed parameter, η, due to the lack of $\rho I\Omega^2\theta$ term in the equations of Lee and Kuo (1993).

Additionally, variation of the fundamental natural frequency of a rotating Timoshenko beam with cantilever end condition with respect to various values of $S(=1/r)$ and η is introduced in Table V. As it is seen in this table, the agreement between the results of the present study and Banerjee (2001) is excellent. In the case of a nonrotating beam $(\eta=0)$, a good agreement with Lee and Kuo (1993) is observed, but in the case of rotation $(\eta=5)$, the results do not match due to the reason mention before.

Moreover, in Table VI, variation of the fundamental natural frequency of rotating Timoshenko beam is given for various values of the Timoshenko effect parameter, r, and the rotational speed parameter, η. As expected, the values of the natural frequencies increase with the increasing rotational speed parameter due to the stiffening effect of the centrifugal force and the natural frequencies decrease as the Timoshenko effect is increased because the shear deformation has a decreasing effect on the natural frequencies. The results of the present study and Banerjee (2001) agree completely. However, due to missing term mentioned before, the difference between Du *et al.* (1994) increases with the increasing rotation speed parameter η.

In Figure 3, variation of the first five natural frequencies of a rotating beam with respect to the Timoshenko effect parameter, r, is given. For all modes, the natural frequencies decrease with increasing r because shear deformation has a decreasing effect on the natural frequencies, but the Timoshenko effect is more dominant on higher modes as expected so the higher mode frequencies of the rotating beam decrease remarkably on account of the rotary inertia parameter while the lower modes are nearly unaffected.

Furthermore, nondimensional frequency variation with respect to the Timoshenko effect, r, is given in Figure 4. Nondimensionalization is made with respect to natural frequency parameter of Bernoulli-Euler beam, μ_0. As discussed above, decreasing of natural frequency for higher modes is evident.

Table IV Variation of the fundamental natural frequencies of a rotating Timoshenko beam with cantilever end condition for various values of the rotational speed parameter η ($\delta = 0$, $r = 1/30$, $E/kG = 3.059$)

	Fundamental natural frequency μ_1		
η	Present study	Banerjee (2001)	Lee and Kuo (1993)
0	3.4798	3.4798	3.4798
1	3.6445	3.6445	3.6452
2	4.0971	4.0971	4.0994
3	4.7516	4.7516	4.7558
4	5.5314	5.5314	5.5375
5	6.3858	6.3858	6.3934
10	11.0643	–	–

Table V Variation of the fundamental natural frequency of a rotating Timoshenko beam with cantilever end condition with respect to the inverse of the Timoshenko effect parameter, $S(= 1/r)$, and the rotational speed parameter, η ($\delta = 0$, $E/kG = 3.059$)

	Fundamental natural frequency parameter (μ)					
	$\eta = 0$			$\eta = 5$		
S	Present study	Banerjee	Lee Kuo	Present study	Banerjee	Lee Kuo
20	3.4364	3.4364	3.4364	6.3126	6.3126	6.3241
30	3.4798	3.4798	3.5798	6.3858	6.3858	6.3934
40	3.4955	3.4955	3.4954	6.4131	6.4131	6.4179
50	3.5028	3.5028	3.5028	6.4260	6.4260	6.4294
80	3.5108	3.5108	3.5108	6.4403	6.4403	6.4418
100	3.5127	3.5127	3.5126	6.4436	6.4436	6.4446
150	3.5145	3.5145	3.5144	6.4469	6.4469	6.4476
200	3.5152	3.5152	3.5152	6.4481	6.4481	6.4485
300	3.5156	3.5156	3.5155	6.4489	6.4489	6.4493
500	3.5159	–	–	6.4493	–	–
1,000	3.5160	–	–	6.4495	–	–

Table VI Variaton of the fundamental natural frequency of a rotating Timoshenko beam with cantilever end condition with respect to the Timoshenko effect r and the rotational speed parameter η ($\delta = 0$, $k = 2/3$, $E/G = 8/3$)

	Fundamental natural frequency parameter (μ)											
	$\eta = 0$			$\eta = 4$			$\eta = 8$			$\eta = 12$		
r	Present study	Banerjee	Du *et al.*	Present study	Banerjee	Du *et al.*	Present study	Banerjee	Du *et al.*	Present study	Banerjee	Du *et al.*
0	3.5160	3.5160	3.516	5.5850	5.5850	5.585	9.2568	9.2568	9.257	13.170	13.170	13.170
0.01	3.5119	3.5119	3.512	5.5791	5.5791	5.580	9.2447	9.2447	9.246	13.148	13.148	13.150
0.02	3.4998	3.4998	3.500	5.5616	5.5616	5.564	9.2096	9.2096	9.215	13.087	13.087	13.087
0.03	3.4799	3.4799	3.480	5.5332	5.5332	5.539	9.1549	9.1549	9.167	12.998	12.998	13.015
0.04	3.4527	3.4527	3.453	5.4951	5.4951	5.505	9.0854	9.0854	9.106	12.893	12.893	12.923
0.05	3.4187	3.4187	3.419	5.4487	5.4487	5.463	9.0060	9.0060	9.036	12.783	12.783	12.827
0.06	3.3787	3.3787	3.379	5.3954	5.3954	5.415	8.9208	8.9208	8.963	12.672	12.672	12.734
0.07	3.3335	3.3335	3.333	5.3370	5.3370	5.363	8.8333	8.8333	8.889	12.564	12.564	12.646
0.08	3.2837	3.2837	3.248	5.2749	5.2749	5.307	8.7456	8.7456	8.815	12.458	12.458	12.564
0.09	3.2302	3.2302	3.230	5.2104	5.2104	5.249	8.6588	8.6588	8.744	12.353	12.353	12.487
0.1	3.1738	3.1738	3.174	5.1448	5.1448	5.191	8.5735	8.5735	8.677	12.247	12.247	12.415
0.15	2.8692	–	–	4.8262	–	–	8.1406	–	–	11.398	–	–

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Figure 3 The first five natural frequencies of a rotating Timoshenko beam

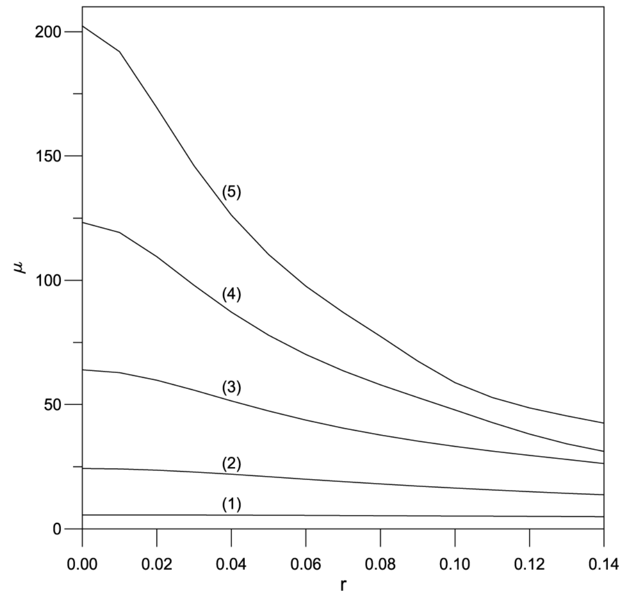
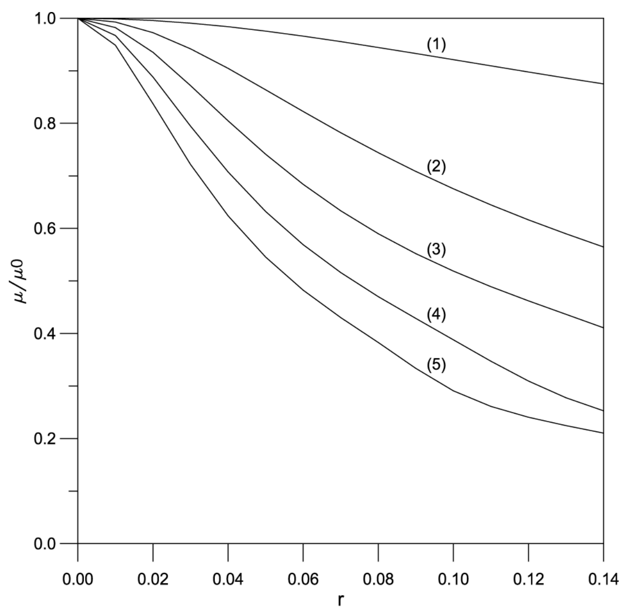


Figure 4 The first five natural frequencies of a rotating Timoshenko beam ($\eta = 4$) (μ_0 corresponds to the natural frequencies of Bernoulli-Euler beam)



The variation of first five natural frequencies with respect to rotational speed, η, is given in Figure 5. As expected, due to the centrifugal stiffening effect, natural frequencies increase with increasing rotational speed, η. The natural frequency increase due to the centrifugal stiffening is much evident for lower modes than the higher ones.

Moreover, variation of the natural frequencies with respect to the hub radius parameter, δ, is introduced in Figure 6. The hub radius has an increasing effect on the value of the centrifugal force so the hub radius parameter has an increasing effect on the natural frequencies.

7. Conclusions

A new and semi-analytical technique called the DTM is applied to the problem of a rotating Timoshenko beam in a simple and accurate way, the natural frequencies are calculated and related graphics are plotted. The effects of the hub radius, rotary inertia, shear deformation and rotational speed are investigated. The numerical results indicate that the natural frequencies increase with the rotational speed and hub radius while they decrease with the rotary inertia (and shear deformation). The effect

Figure 5 The effect of rotational speed on the first five natural frequencies of a rotating Timoshenko beam ($r = 0.04$)

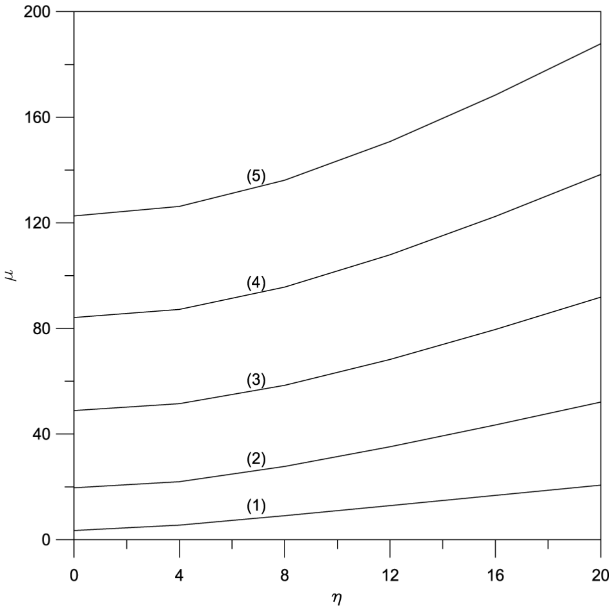
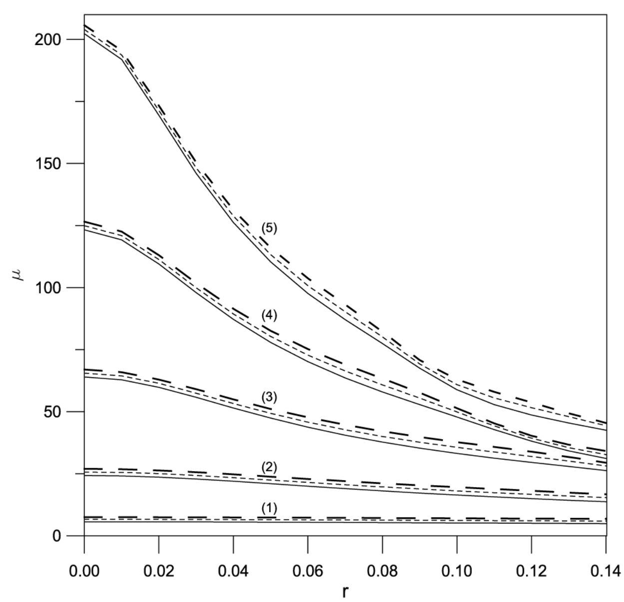


Figure 6 Effect of hub radius on the first five natural frequencies of a rotating Timoshenko beam



Notes: $\eta = 4$; $\delta = 0$; ———, $\delta = 0.5$; - - - - -, $\delta = 1.0$ — —

of the rotary inertia (and shear deformation) is more dominant on the higher modes. The calculated results are compared with the ones in literature and great agreement is considered.

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Further reading

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